PROPAGATION AND REFLECTION OF TORSIONAL WAVES IN SPHERICAL BODIES

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1. Introduction

In the presence of a background magnetic field, a conducting fluid can sustain magneto-hydrodynamic waves, known as Alfvén waves. In rapidly rotating axisymmetric bodies the dominance of geostrophic effects gives rise to torsional waves (TW). These are axisymmetric oscillations of the azimuthal velocity and magnetic field whose restoring force is set by a background magnetic field perpendicular to the rotation axis. Both the background field and the wave amplitude are function of the cylindrical radius *s* only. On Earth, the detection of the fundamental period of the TW may provide a direct way to assess the shape and the magnitude of the background magnetic field.
Attempts in detecting TW from observations (Hide et al., 2000 and Gillet et al., 2010) suggest the absence of reflections at both the Core Mantle Boundary (CMB) and the rotation axis, a feature that has not yet been explained.

In particular Gillet et al., 2010 detected axisymmetric oscillations with periodicity of 6 to 8 years, amplitudes of 0.4 Km yr^{-1} and travelling from the Inner Core Boundary to the (CMB) in 4 years. TW theory predicts a background field as big as 3 mT in the interior of the liquid core.
Here we want to study the propagation of TW in spherical domains, paying particular attention to the reflections and boundary conditions.
We will study normal modes and the propagation of an initially concentrated pulse across the domain with different background fields, as also done in Cox et al., 2013. We then illustrate how the WKBJ approximation could be used to derive the reflection coefficients at the boundaries. We apply this technique to the case of a newly derived closed form solution to the TW equation.

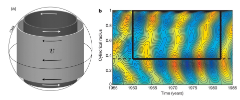


Figure 1: (left) From Roberts and Aurnou, 2012: schematics of TWs in the core. (right) From Gillet et al., 2010: TW time evolution derived from the data. Blue and red are negative and positive angular velocities.

2. Normal modes

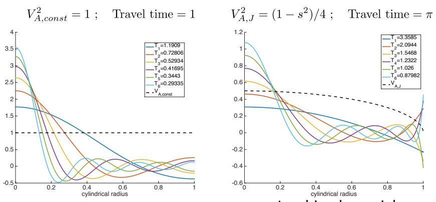
We solve the diffusion-free TW normal mode equation using the finite element solver Comsol

$$-\omega_n^2 m\zeta_n = \frac{d}{ds}\left[mV_A^2\frac{d\zeta_n}{ds}\right] \quad ; \qquad m = s^3H$$

$$\zeta = \frac{v}{s} \quad ; \qquad V_A^2 = \frac{\langle B_{s,0}^2\rangle}{\mu_0\rho_0}$$

Here s is the cylindrical radius and the boundary conditions for the full sphere are:

- Regularity condition at *s*=0 : $\zeta'|_{s=0} = 0$
- Insulating BC at *s*=1: $\zeta'|_{s=1} = 0$ if $V_A^2 \neq 0$



Oblate spheroidal wave functions (Roberts and Aurnou, 2012)

Jacobi polynomials

$$\zeta_n = P_n^{(1/2,1)}(2s^2-1)$$

$$\omega_n = \sqrt{n(n+5/2)}$$

Figure 2: Eigenmodes and eigenperiods calculated for a constant (left) and what we will call the Jacobi (right) Alfvèn velocity profiles. The dotted line is the Alfvèn velocity.

- The travel time is always slightly shorter than the first eigenperiod
- The Jacobi field solutions satisfy insulating BCs even if the derivative is non-zero at the equator

3. Time evolution

Next, we integrate the diffusion-free TW wave equation forward in time using the finite element solver Comsol (low order BDF method)

$$s^3H\frac{\partial^2\zeta}{\partial t^2} = \frac{\partial}{\partial s}\left[s^3HV_A^2\frac{\partial\zeta}{\partial s}\right]$$

We obtain the same result by projecting the initial condition on the normal modes basis and evolving them in time according to their eigenfrequencies.

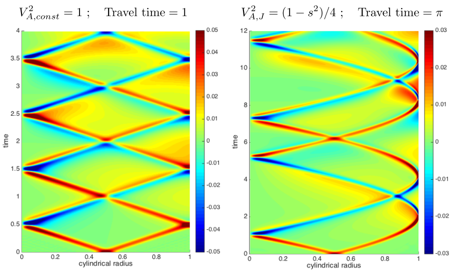


Figure 3: Time evolution of azimuthal velocity from a pulse initially focalized in *s*=0.5 for a constant (left) and Jacobi (right) Alfvén velocity.

- The pseudo-reflection at the rotation axis is similar in both cases, as the Alfvén velocity has to satisfy the same regularity condition
- The reflection at the equator depends on the value of the Alfvén velocity there

4. WKBJ approximation

It is possible to approximate both the normal mode solution and a propagating wave with a first order WKBJ expansion (Bender and Orszag, 1999)

$$\zeta_{WKBJ}(s,t) = \exp[\omega S_0(s) + S_1(s) - \mathrm{i}\omega t]$$

$$= \sqrt{\frac{1}{mV_A}}\exp\left[\pm\mathrm{i}\omega\int\frac{ds}{V_A} - \mathrm{i}\omega t\right]$$

The approximation is valid in the interior of the domain: it comprises an oscillatory part and a modulation.

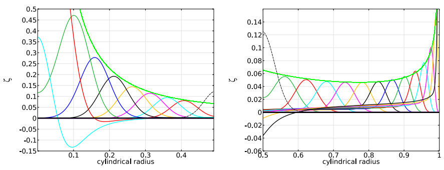


Figure 4: Modulating factor in the WKBJ expansion (green dashed line) for the Jacobi field compared to successive snapshots from Figure 3, before the first reflections. The initial condition is the black dashed-dotted line.

To have a solution valid in the whole domain we need to calculate boundary solutions to the TW:

- ζ_I : boundary solution valid for $s \ll 1$
- ζ_{III} : boundary solution valid for $(1-s) \ll 1$

The matching of the WKBJ solution with these boundary solutions gives the approximated eigenmodes and eigenfrequencies

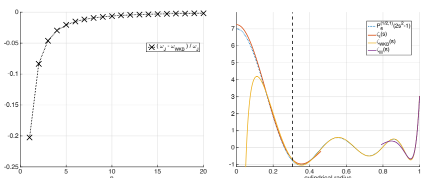


Figure 5: (left) true eigenfrequencies compared with the WKBJ approximation for the Jacobi field. Note that the error is less than 21 % even for the first mode. (right) sixth Jacobi eigenmode compared with the WKBJ approximation and the boundary solutions.

5. Reflections

If we consider the temporal dependence in the WKBJ solution, we can study reflections of wave-like solutions. The matching with the boundary solutions gives imaginary reflection coefficients that we can express in terms of phase shifts. For the Jacobi solution:

Rotation axis (s=0) $\quad \Delta\mu_0 = -\frac{3}{2}\pi \quad$ frequency independent

Equator (s=1) $\quad \Delta\mu_1 = 2\pi\omega + \pi \quad$ frequency dependent

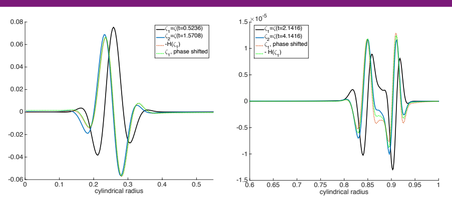


Figure 6: Predicted and true reflections at the rotation axis (left) and equator (right) for the Jacobi velocity profile and an initial condition enriched in high frequencies and centred on *s*=0.5 (left) and *s*=0 (right). First reflection only.

6. Conclusions

We present a new analytical solution, in terms of Jacobi polynomials, for the TW problem in a full sphere with vanishing Alfvén velocity at the equatorial boundary. We propose a method based on the WKBJ approximation to get analytic insights on the TW eigenmodes and reflections at the boundaries. We applied this method on the Jacobi solution. The reflection coefficients at both boundaries are imaginary. At the rotation axis a frequency independent phase shift is introduced that resembles the phase shift introduced on a seismic wave passing trough a caustic (Chapman, 2004) and corresponds to the action of a Hilbert transform on the incoming wave. At the equatorial boundary the phase shift is frequency dependent.

Given the regularity condition on the Alfvén velocity at the rotation axis, we expect the results derived here to be of general character, that is, they can be applied to other velocity profiles. The reflection at the equatorial boundary, however, strongly depends on the velocity field.
Gillet et al., 2010 commented that absence of reflections could be explained by strong gradient in the Alfvén velocity next to the boundaries or by finite conductivity at the CMB, which would result in introducing diffusivity in the TW equation. Since we find non-zero reflections even for high gradients velocity profiles, the absence of reflections could be explained considering finite CMB conductivity.

References

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