Mean fields versus local fields in convection dynamos

Andreas Tilgner

Georg-August-Universität Göttingen

Rotating convection in the Boussinesq approximation

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + 2\frac{Pr}{Ek}\hat{\boldsymbol{z}} \times \boldsymbol{v} = -\nabla p + Pr \ \nabla^2 \boldsymbol{v} + Ra \ Pr \ T\hat{\boldsymbol{z}}$$

$$\nabla \cdot \boldsymbol{v} = 0$$

$$\partial_t T + \boldsymbol{v} \cdot \nabla T = \nabla^2 T$$

Control parameters:

$$Ra = \frac{g\alpha\Delta T d^3}{\kappa\nu} \quad , \quad Ek = \frac{\nu}{\Omega d^2} \quad , \quad Pr = \frac{\nu}{\kappa}$$

Boundary conditions:

$$T(z=0)=1 \quad , \quad T(z=1)=0 \quad \text{fixed temperature}$$

$$v_z = \partial_z v_x = \partial_z v_y = 0 \quad \text{free slip}$$

periodic lateral boundary conditions

Output parameters:

$$Re = \frac{1}{Pr}\sqrt{\frac{1}{V}\int < \boldsymbol{v}^2 > dV} \quad , \quad Nu = -\frac{1}{A}\int < \partial_z T > dA$$

The Nusselt number

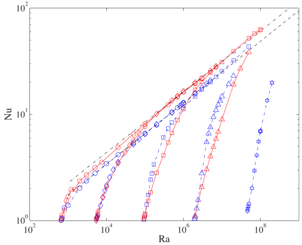


Figure: Nu as a function of Ra for $Pr = 7$ (red symbols and continuous line) and 0.7 (blue symbols and dot dashed line), and $Ek = 2 \times 10^{-2}$ (diamonds), 2×10^{-3} (squares), 2×10^{-4} (triangles) and 2×10^{-5} (stars). Zero rotation is indicated by circles and the power law fits have an exponent of 0.287.

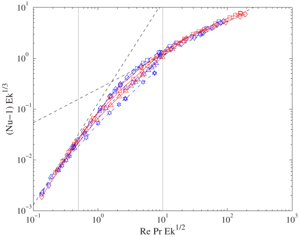


Figure: $(Nu-1)Ek^{1/3}$ as a function of $Re\, Pr\, Ek^{1/2}$ for the same data and with the same symbols as in the left figure. The dashed lines are power laws with exponents 2 and 2/3. The two vertical lines indicate the interval outside of which the fit to one of the two power laws is considered satisfactory.

The dynamo

$$\nabla \cdot \boldsymbol{v} = 0$$

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + 2\frac{\mathrm{Pr}}{\mathrm{Ek}}\hat{\boldsymbol{z}} \times \boldsymbol{v} =$$

$$-\nabla p + \mathrm{Pr}\ \mathrm{Ra}\ \theta\hat{\boldsymbol{z}} + \mathrm{Pr}\nabla^2\boldsymbol{v} + (\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$$

$$\partial_t \theta + \boldsymbol{v} \cdot \nabla\theta - v_z = \nabla^2\theta$$

$$\partial_t \boldsymbol{B} + \nabla \times (\boldsymbol{B} \times \boldsymbol{v}) = \frac{\mathrm{Pr}}{\mathrm{Pm}}\nabla^2\boldsymbol{B}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

Control parameters:

$$\mathrm{Ra} = \frac{g\alpha\Delta T d^3}{\kappa\nu} \quad , \quad \mathrm{Ek} = \frac{\nu}{\Omega d^2} \quad , \quad \mathrm{Pr} = \frac{\nu}{\kappa} \quad , \quad \mathrm{Pm} = \frac{\nu}{\lambda}$$

Boundary conditions:

$$T = T_0 + \Delta T \text{ at } z = 0 \quad , \quad T = T_0 \text{ at } z = d,$$

$$v_z = \partial_z v_y = \partial_z v_x = 0 \text{ at } z = 0, d,$$

$$B_z = \partial_z B_y = \partial_z B_x = 0 \text{ at } z = 0, d.$$

Fixed temperature, free slip, perfect conductor
Periodic lateral boundary conditions

The kinematic dynamo

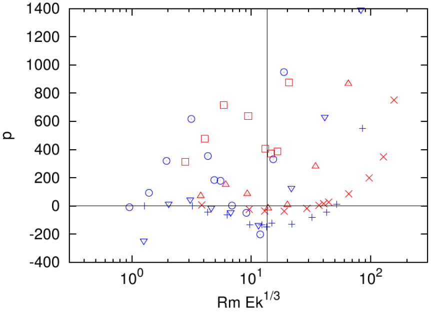


Figure: Growth rate p of magnetic energy in kinematic dynamo calculations as a function of $\mathrm{Rm}\,\mathrm{Ek}^{1/3}$. Results for $Pm = 1$ are shown in blue and those for $Pm = 3$ are in red. For $Pm = 1$, the Ekman numbers of 2×10^{-4}, 2×10^{-5}, and 2×10^{-6} are indicated by the plus sign, triangle down, and circle, respectively, whereas for $Pm = 3$, the same Ekman numbers are indicated by the x sign, triangle up, and square. The vertical line is located at $\mathrm{Rm}\,\mathrm{Ek}^{1/3} = 13.5$.

Small mean field or no mean field?

- Volume V, cross section of every horizontal plane $A \times A$
- Total magnetic energy $E_B = \langle \frac{1}{V} \int \frac{1}{2} \boldsymbol{B}^2 dV \rangle$
- Energy in the mean field
  $\bar{E} = \frac{1}{2} \frac{1}{VA^2} < \int dz \left(\int dy \int dx \, \boldsymbol{B} \right)^2 >$
- $\bar{E}$ is never zero because of random fluctuations in the magnetic field.
- Magnetic field with correlation length l_c.
- Number of independent degrees of freedom in a plane with cross secion $A \times A$: $(A/l_c)^2$.
- Compute the mean field in a plane:
  - Add $(A/l_c)^2$ numbers drawn from a probability distribution of width $\sqrt{E_B}$.
  - Divide the sum by the number of samples $(A/l_c)^2$.
  - The square of this average is $\left(\sqrt{(A/l_c)^2} \sqrt{E_B} (A/l_c)^2 \right)^2$.
- $\bar{E} \propto (l_c/A)^2 E_B$

Small mean field or no mean field?

- $\bar{E}/(E_B - \bar{E}) \approx \bar{E}/E_B \propto (l_c/A)^2$

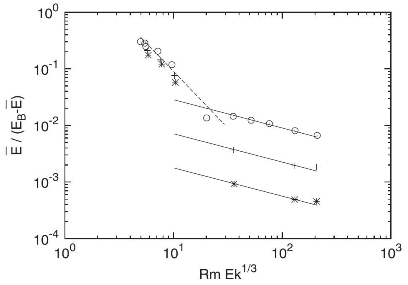
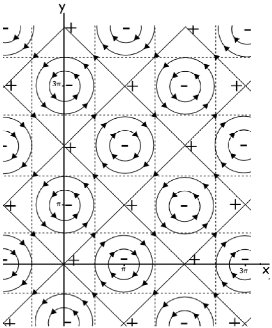
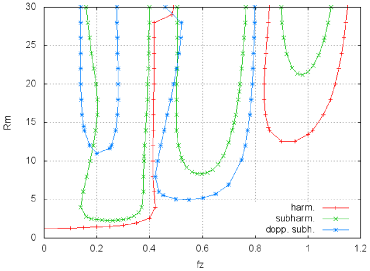


FIG. 5. $\bar{E}/(E_B - \bar{E})$ as a function of $\mathrm{Rm\,Ek}^{1/3}$ for $A = 0.5$ (circles), 1 (crosses), and 2 (stars). The dashed line follows the prediction of first order smoothing and shows $\bar{E}/(E_B - \bar{E}) \propto (\mathrm{Rm\,Ek}^{1/3})^{-2}$. The solid lines plot the functions $0.09/\sqrt{x}$, $0.09/(4\sqrt{x})$, and $0.09/(16\sqrt{x})$.

G.O. Roberts (Flow III)



G.O. Roberts (1972)



Johannes Teichmann, Master Thesis (2015)

Conclusion

- Two different types of dynamos exist within a hydrodynamically uniform regime of Rayleigh-Bénard convection.
- At the transition, the magnetic Reynolds number based on the diameter of a vortex is 26.
- No mean magnetic field is generated above the transition.
- One has to simulate different aspect ratios to show that there is no mean field.
- Mean fields can disappear from periodic dynamos after a mere change of a peiodicity length.